STEM Games Day1

May 25, 2022

1 Introduction

In all sorts of production as well as maintenance of infrastructure it is often important to measure the quality and integrity of products or their parts. The simplest way to do this is to take apart a small percentage of products and check them for flaws. While simple, this method is also wasteful. To combat this wastefulness, many techniques of non destructive testing have been developed. However, these techniques rely on complicated mathematical tools to work properly, and without proper mathematical analysis are often extremely susceptible to noise. Today, we will focus on the problem of convolution, which models blurring that happens inside sensors, and the problem of deconvolution.

1.1 Problem: Convolution

Definition 1.1 *Let $I = \langle a, b\rangle \subseteq \mathbb{R}$ be an interval. The convolution of the functions $f, g : I \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ defined as:*

$$(f * g)(x) := \int_I \tilde{f}(t)\tilde{g}(x-t)dt, \quad \forall x \in \mathbb{R}, \tag{1}$$

Where $\tilde{f}, \tilde{g}$ are f, g expanded with zero values.

One could interpret convolution of f and g at the x to be the area under the intersection of the graphs of f and reversed g function. For the visual representation please visit following links:

- https://en.wikipedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif
- https://en.wikipedia.org/wiki/File:Convolution_of_spiky_function_with_box2.gif

Task 1.2 *Let $f : [0, 1] \to \mathbb{R}$, $f(x) = 1$. Determine $f * f$.*

Task 1.3 *Let f and g be some functions such that $f, g : I \to \mathbb{R}$ where $I = \langle a, b\rangle \subseteq \mathbb{R}$. Prove the following properties of convolution:*

a) Associativity

b) Does not have multiplicative identity *c) Each function has inverse element*

d) Distributivity

e) Commutativity

Task 1.4 *For the given function f and g, calculate $f * g$ and roughly sketch it. HINT: To help yourself with visualisation, sketch the functions f and g as well.*

a) $f(x) = x^2$ *and* $g(x) = \mathbb{1}_{[-1,1]}$

b) $f(x) = x$ *and* $g(x) = \sqrt{x} \cdot \mathbb{1}_{[0,3]}$

c) $f(x) = sgn(cos(8\pi x))$ *and* $g(x) = x \cdot \mathbb{1}_{[-2.2]}$

Definition 1.5 *Let $I = \{-M, -M+1, \dots, M-1, M\}$ where $M \in \mathbb{Z}$. For functions f and g we define **discrete convolution** on a give set as:*

$$(f * g)[n] = \sum_{k=-M}^{M} f(n)g(n-k)$$

Note that the definition is easily extended on any discrete set.

Task 1.6 *Repeat the task 1.4, but this time convert each of the intervals into a discrete set. Compare the values of discrete convolution and convolution. Make a discrete set from the interval by dividing it into 50 parts.*

1.2 Problem: Least squares deconvolution

The problem of deconvolution can be stated as follows: If we assume that the output of the system is convolved with a known function, find the input signal. Assume that the output is given by:

$$y(n) = h(0)x(n) + h(1)x(n-1) + \cdots + h(N)x(n-N), \tag{2}$$

where x(n) is the input signal, and h(n) is a known function. Equation 2 can be written as $\mathbf{y} = \mathbf{Hx}$. Here $\mathbf{H}$ is a symmetric Toeplitz matrix. **Task**: retrive x by solving the linear system

$$Hx = y$$

where H is a symmetric band diagonal matrix with $\frac{1}{2}$ on its diagonal, $\frac{1}{3}$ on its first off diagonal and $\frac{1}{6}$ on its second off diagonal. For right hand side take the vector from *konv0.txt*. It is created from a known vector by convolution and then adding noise sampled from normal distriudion with mean 0 and standard deviation 0.05.

Compare the solution to the solution of the minimization problem:

$$\min_x \|\mathbf{y} - \mathbf{Hx}\|^2 + \lambda \|x\|^2 \tag{3}$$

Find this solution for $\lambda = 10^{-1}$, $\lambda = 1$ and $\lambda = 10$.

Compare all of these solutions to the ground truth solution in *signal1.txt*.

1.3 Regularization

$$\min_x \|\mathbf{y} - \mathbf{Hx}\|^2 + R(x) \tag{4}$$

We have seen that solving

$$\min_x ||Ax - b||_2^2 + \lambda ||x||_2^2$$

Let us now prove some properties of this formulation of problem. Assuming that the matrix A is regular, the problem $Ax = b$ has a unique solution. Let us define $H(\lambda) = x$ as a function that maps λ to the x that minimizes

$$||Ax - b||_2^2 + \lambda ||x||_2^2$$

Prove that this function is well defined. In other words

- Prove that that for each $\lambda > 0$ there exists exactly one x. (Assuming that A is a full rank matrix)

Alternatively, we can write this idea in a different way

$$\min_x ||Ax - b|| \textbf{ s.t. } ||x||_2^2 \leq \varepsilon$$

Task 1.7 *Prove that for every choice of ε there exists λ such that the minimizer of one problem is the same as minimizer of the other problem.*

We can also define H in a more general way, as the minimizer of the expression

$$\min_x ||Ax - b||_2^2 + \lambda R(x)$$

Task 1.8

- *Prove that a solution exists and is unique for $R(x) = ||Dx||_2^2$, where D is an arbitrary regular linear operator.*
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- *Prove that a solution exists and is unique for $R(x) = ||x||_1$*
- *Prove that a solution exists and is unique for $R(x) = ||Dx||_1$, where D is an arbitrary linear operator.*

PBZ task:

Another important place to use regularization is in machine learning. Linear least squares problems also take the form of $\min ||Ax - b||_2^2$, and in presence of noise in b, regularization can improve the result. It can be useful to explore how the solution of the regularized problem behaves as a function of the parameter λ.

Task 1.9 *If the columns of matrix A are orthonormal, show that the solution of*

$$\min ||Ax - b||_2^2$$

and of

$$||Ax - b||_2^2 + \lambda ||x||_2^2$$

are colinear.

Task 1.10

- *Using file Konv1.txt try to reconstruct the function from file Signal1 asuming it was convolved with a Gaussian wit mean of* 0 *and standard deviation of* 0.1*. The result also has noise added to it taken from a normal distriution with mean* 0 *and standard deviation* 0.1*. Do this using regularization based on* $R(x) = ||x||_2^2$, $R(x) = ||\nabla x||_2^2$, $R(x) = ||x||_1$ *and* $R(x) = ||\nabla x||_1$*. Here* ∇ *is a discrete approximation of the derivative operator.*

- *Using file Konv2.txt try to reconstruct the function from file Signal2 asuming it was convolved with a Gaussian wit mean of* 0 *and standard deviation of* 0.1*. The result also has noise added to it taken from a normal distriution with mean* 0 *and standard deviation* 0.05*. Do this using regularization based on* $R(x) = ||x||_2^2$, $R(x) = ||\nabla x||_2^2$, $R(x) = ||x||_1$ *and* $R(x) = ||\nabla x||_1$.

- *Using file Konv3.txt try to reconstruct the function from file Signal1 asuming it was convolved with the vector* $[0.125, 0.125, 0.25, 0.5, 0.25, 0.125, 0.125]$*. The result also has noise added to it taken from a normal distriution with mean* 0 *and standard deviation* 0.1*. Do this using regularization based on* $R(x) = ||x||_2^2$, $R(x) = ||\nabla x||_2^2$, $R(x) = ||x||_1$ *and* $R(x) = ||\nabla x||_1$.

Compare all these solutions to unrerularized solutions.

1.4 Projectional methods

More programming oriented approach

Instead of solving the problem $min||Ax - b||$, the idea is to solve the projected problem $min||AVc - b||$ where $x = Vc$, and columns of matrix V are the orthonormal basis for some space where we want to look for the solution. In general we split between fixed subspaces and problem specific subspaces.

We can subdivide the interval $[0, 1]$ into $n + 1$ equal parts so we can approximate a function on that interval using finite dimensional vectors.

Let us define

$$v_{k,n}^{(i)} = \cos\left(k \frac{2i + 1}{2n} \pi\right)$$

In other words i-th (0 indexed) component of the k-th vector when subdivided into n parts is $\cos\left(k \frac{i}{n} 2\pi\right)$. This can be understood as a discretization of the function $\cos\left(k \frac{x}{n} 2\pi\right)$ into $n + 1$ points on the interval $[0, 1]$.

- Prove that for any n the set of vectors $v_{k,n}$ $k \in \{0, 1, .., n - 1\}$ is an orthogonal basis.

Task 1.11

- *Find projections of noisy signal1.txt and noisy signal2.txt onto the subspace spanned by first k discrete cosines for $k = 5$, $k = 10$ and $k = 20$. Compare them to original signal1 and signal2*
- *Deconvolve konv1, konv2 and konv3 from before by using projection onto space spanned by first k discrete cosines for $k = 5$, $k = 10$ and $k = 20$.*